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LETTER TO THE EDITOR

On the point spectrum of the N-points Friedrichs model

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Abstract. We consider the N-points Friedrichs ($N>1$) model and show, by means of a counterexample, that a theorem of Marchand about the non-existence of point eigenvalues embedded in the continuous spectrum is incorrect.

We consider the N-points Friedrichs (1948) model. The spectrum of the 'unperturbed' Hamiltonian, H_0, for this model consists in a continuous part extending over some interval $\mathscr{I}$ of the real axis and a point spectrum with a finite number of eigenvalues (ω_i^0; $i=1,\ldots,N$) embedded in the continuum. An interaction λV (λ being a real coupling constant) couples the point spectrum to the continuum.

In the spectral representation of H_0, the 'perturbed' Hamiltonian $H=H_0+\lambda V$ is given by a 'matrix' (Grecos 1978):

$$H \Leftrightarrow \begin{bmatrix} \omega_1^0 & & 0 & \lambda v_1(\omega') \\ & \ddots & & \vdots \\ 0 & & \omega_N^0 & \lambda v_N(\omega') \\ \lambda v_1(\omega) & \ldots & \lambda v_N(\omega) & \omega\delta(\omega-\omega') \end{bmatrix} \tag{1}$$

and an element 'f' of the Hilbert space $\mathscr{H}$ on which H acts is represented by a column vector:

$$f \Leftrightarrow \{f_1,\ldots,f_N, f(\omega)\}. \tag{2}$$

The f_i's are complex numbers and $f(\omega)$ is a square integrable function on $\mathscr{I}$. The Hilbert space $\mathscr{H}$ is equipped with the scalar product

$$\langle f, g\rangle = \sum_{i=1}^{N} \bar{f}_i g_i + \int_{\mathscr{I}} \mathrm{d}\omega' \, \bar{f}(\omega') g(\omega') \tag{3}$$

and the element 'Hf' is represented by

$$Hf \Leftrightarrow \{\omega_1^0 f_1 + \lambda \int_{\mathscr{I}} \mathrm{d}\omega \, \bar{v}_1(\omega) f(\omega), \ldots, \omega_N^0 f_N + \lambda \int_{\mathscr{I}} \mathrm{d}\omega \, \bar{v}_N(\omega) f(\omega), \lambda \sum_{i=1}^{N} v_i(\omega) f_i + \omega f(\omega)\}. \tag{4}$$

For the one-point Friedrichs model ($N=1$) it has often been shown (Friedrichs 1948, Marchand 1967) that there does not exist any discrete eigenvalue of H embedded in the continuum if we assume that $v_1(\omega)$ does not vanish on $\mathscr{I}$. An extension of this theorem for the N-points Friedrichs model, when $N>1$, was given by Marchand

(1964). The purpose of this letter is to show, by means of a counterexample, that Marchand's theorem about the absence of eigenvalues embedded in the continuum is incorrect.

According to Marchand (1964) we assume that

(i) $v_i(\omega) \neq 0$ for all $\omega \in \mathscr{I}$ and for all $i = 1, \ldots, N$ except at the end-point(s) of $\mathscr{I}$. In order to exclude some trivial cases, we assume also that

(ii) $v_i(\omega) \neq v_j(\omega)$ for almost all $\omega \in \mathscr{I}$ and for all $i \neq j$.

We remark that, without any loss of generality, one may choose an ordering $i = 1, \ldots, N$ such that $v_N(\omega)/v_i(\omega)$ be finite at the end-point(s) of $\mathscr{I}$; for $\omega \in \mathscr{I}$, these quantities are finite in virtue of assumption (i). Therefore, one may expand $v_N(\omega)$ in terms of the other $v_i(\omega)$ as follows:

$$v_N(\omega) = \sum_{i=1}^{N-1} w_i(\omega) v_i(\omega) \tag{5}$$

where the $w_i(\omega)$ are given by ($i = 1, \ldots, N-1$)

$$w_i(\omega) = v_N(\omega)/(N-1)v_i(\omega). \tag{6}$$

Let us now consider the eigenvalue problem for H:

$$(\omega_k^0 - \nu) f_k + \lambda \int_{\mathscr{I}} \mathrm{d}\omega' \, \bar{v}_k(\omega') f(\omega') f(\omega') = 0 \qquad k = 1, \ldots, N \tag{7}$$

$$\lambda \sum_{j=1}^{N} v_j(\omega) f_j + (\omega - \nu) f(\omega) = 0 \tag{8}$$

and let us assume the point eigenvalue, ν, to be embedded in the continuum, i.e. $\nu \in \mathscr{I}$. Following Marchand one has

$$f(\omega) = -\lambda \sum_{j=1}^{N} v_j(\omega) f_j/(\omega - \nu) \qquad \omega \neq \nu \tag{9}$$

but from equation (8) one also gets (putting $\omega = \nu$)

$$\lambda \sum_{j=1}^{N} v_j(\nu) f_j = 0 \tag{10}$$

so that, instead of equation (9), we have

$$f(\omega) = -\lambda \sum_{j=1}^{N} \{[v_j(\omega) - v_j(\nu)]/(\omega - \nu)\} f_j. \tag{11}$$

Equation (11) is defined even for $\omega = \nu$ and it is a square integrable function. Marchand's theorem about the absence of eigenvalues (of H) embedded in the continuum is, therefore, wrong as we shall see by means of a counterexample.

Before going on to this counterexample, let us rewrite (11) in a more appropriate form. Using (5) we rewrite (8) and (10) as follows

$$\lambda \sum_{j=1}^{N-1} v_j(\omega)(f_j + w_j(\omega) f_N) + (\omega - \nu) f(\omega) = 0 \tag{12}$$

$$\lambda \sum_{j=1}^{N-1} v_j(\nu)(f_j + w_j(\nu) f_N) = 0. \tag{13}$$

Equation (13) admits a trivial solution

$$f_j = -w_j(\nu) f_N \qquad j = 1, \ldots, N-1 \tag{14}$$

which is, by assumption (i), the only solution in the case $N = 2$ (two-points Friedrichs model). Using (12) and (14) we get

$$f(\omega) = -\lambda \sum_{j=1}^{N-1} v_j(\omega) \frac{w_j(\omega) - w_j(\nu)}{\omega - \nu} f_N. \tag{15}$$

The coefficient f_N is fixed by the normalisation condition. Insertion of (14) and (15) into the left-hand side of (7) gives

$$(\omega_k^0 - \nu) w_k(\nu) + \lambda^2 \sum_{j=1}^{N-1} \hat{\sigma}_{kj}(\nu) = 0 \qquad k = 1, \ldots, N-1 \tag{16}$$

$$\omega_N^0 - \nu - \lambda^2 \sum_{j=1}^{N-1} \hat{\sigma}_{Nj}(\nu) = 0 \tag{17}$$

with

$$\hat{\sigma}_{kj}(\nu) = \int_{\mathscr{I}} \mathrm{d}\omega\, \bar{v}_k(\omega) v_j(\omega) \frac{w_j(\omega) - w_j(\nu)}{\omega - \nu}. \tag{18}$$

The eigenvalues embedded in the continuum are given by the solutions of (16) and (17) and we will show now that for some choices of $(v_i(\omega);\ i = 1, \ldots, N)$ these equations admit a solution $\nu \in \mathscr{I}$ to which corresponds a finite and real value for the coupling parameter λ.

Example 1. We consider the two-points Friedrichs model ($N = 2$) and we choose $w_1(\omega) = \omega$ and $\mathscr{I} = [0, +\infty)$. Assumptions (i) and (ii) are satisfied if we assume that $v_1(\omega) \neq 0$ for all $\omega \in \mathscr{I}$. Equations (16) and (17) become

$$(\omega_1^0 - \nu)\nu + \lambda^2 n_0 = 0 \tag{19}$$

$$\omega_2^0 - \nu - \lambda^2 n_1 = 0 \tag{20}$$

with

$$n_k = \int_0^{+\infty} \mathrm{d}\omega\, \omega^k |v_1(\omega)|^2 \qquad k = 0, 1. \tag{21}$$

Equations (19) and (20) admit two solutions

$$2\nu_\pm = (\omega_1^0 - n_0/n_1) \pm [\omega_1^0 - n_0/n_1)^2 + 4\omega_2^0 n_0/n_1]^{1/2} \tag{22}$$

to which correspond respectively

$$\lambda_\pm^2 = (\omega_2^0 - \nu_\pm)/n_1. \tag{23}$$

By assumption $\omega_2^0 > 0$ and thus $\nu_+ > 0$, $\nu_- < 0$. If $\omega_2^0 > \omega_1^0$, then $\omega_1^0 < \nu_+ < \omega_2^0$ and $\lambda_\pm^2 > 0$. It is thus not necessary to require that ω_1^0 also be embedded in the continuum. Indeed, if $\omega_1^0 = -n_0/n_1$ and $\omega_2^0 = 3n_0/n_1$, then $\nu_+ = n_0/n_1$ with $\lambda_+^2 = 2n_0/n_1$ and $\nu_- = -3n_0/n_1$ with $\lambda_-^2 = 6n_0/n_1$.

Example 2. We consider again a two-points Friedrichs model and we choose

$$v_1(\omega) = \alpha\sqrt{\mu}/(\omega - \omega_1^0 + \mathrm{i}\alpha) \qquad v_2(\omega) = \beta\sqrt{\gamma}/(\omega - \omega_2^0 + \mathrm{i}\beta) \tag{24}$$

and $\mathscr{I} = (-\infty, +\infty)$. Equations (16) and (17) become

$$(\omega_1^0 - \nu)(\nu - \omega_1^0 + \mathrm{i}\alpha)[\omega_1^0 - \omega_2^0 + \mathrm{i}(\alpha+\beta)] + \pi\lambda^2\mu\alpha[\omega_1^0 - \omega_2^0 - \mathrm{i}(\alpha-\beta)] = 0 \tag{25}$$

$$(\omega_2^0 - \nu)(\nu - \omega_2^0 + \mathrm{i}\beta)[\omega_1^0 - \omega_2^0 - \mathrm{i}(\alpha+\beta)] + \pi\lambda^2\gamma\beta[\omega_1^0 - \omega_2^0 - \mathrm{i}(\alpha-\beta)] = 0. \tag{26}$$

Let μ and γ be given by

$$\mu\alpha = (\omega_2^0 - \omega_1^0)^2 - (\alpha^2 - \beta^2) \tag{27}$$

$$\alpha\beta = (\omega_2^0 - \omega_1^0)^2 + (\alpha^2 - \beta^2). \tag{28}$$

Equations (25) and (26) then have one real solution

$$\nu_0 = \omega_1^0 + \frac{(\omega_2^0 - \omega_1^0)^2 - (\alpha^2 - \beta^2)}{2(\omega_2^0 - \omega_1^0)} = \omega_2^0 - \frac{(\omega_2^0 - \omega_1^0)^2 + (\alpha^2 - \beta^2)}{2(\omega_2^0 - \omega_1^0)} \tag{29}$$

to which corresponds

$$\lambda_0^2 = [(\omega_2^0 - \omega_1^0)^2 + (\alpha+\beta)^2]/4\pi(\omega_2^0 - \omega_1^0)^2. \tag{30}$$

It is interesting to note that this solution is related to the existence of the point eigenvalue ω_2^0 of the unperturbed Hamiltonian H_0. We shall illustrate this point by studying the analytic continuation of a dispersion relation.

Indeed we know that the poles of the resolvant of H, i.e. the point spectrum of H, are given by the solutions of a dispersion relation (Grecos 1978, Marchand 1967):

$$0 = \Delta(\nu) = \det(\eta_{kj}(\nu)) \tag{31}$$

with

$$\eta_{kj}(\nu) = (\omega_k^0 - \nu)\delta_{kj} - \lambda^2\sigma_{kj}(\nu) \tag{32}$$

and

$$\sigma_{kj}(\nu) = \int_{\mathscr{I}} \mathrm{d}\omega\, \bar{v}_k(\omega) v_j(\omega)/(\omega - \nu). \tag{33}$$

Here $\delta_{kj} = 1$ if $k = j$ and 0 if $k \neq j$.

For the example under consideration the analytic continuation of the dispersion relation is trivial because the continuous spectrum runs from $-\infty$ to $+\infty$. The analytic continuation from $\operatorname{Im} z > 0$ to $\operatorname{Im} z < 0$ is given by

$$0 = \Delta_+(z) = P_+(z)/[(z - \omega_1^0 + \mathrm{i}\alpha)(z - \omega_2^0 + \mathrm{i}\beta)] \tag{34}$$

with

$$P_+(z) = [(\omega_1^0 - z)(z - \omega_1^0 + \mathrm{i}\alpha) + \pi\lambda^2\mu\alpha][(\omega_2^0 - z)(z - \omega_2^0 + \mathrm{i}\beta) + \pi\lambda^2\gamma\beta] - \frac{4\pi^2\lambda^4\mu\gamma(\alpha\beta)^2}{(\omega_2^0 - \omega_1^0)^2 + (\alpha+\beta)^2}. \tag{35}$$

For $\lambda = 0$, $\Delta_+(z)$ has two real zeros (ω_i^0; $i = 1, 2$). For $\lambda \neq 0$, $\Delta_+(z)$ has four complex roots ($z_i^0(\lambda)$; $i = 1, \ldots, 4$) which tend respectively to ω_1^0, ω_2^0, $-\mathrm{i}\alpha$ and $\omega_2^0 - \mathrm{i}\beta$ at the limit $\lambda \to 0$. The trajectories of these roots, as functions of λ, are shown in figure 1. As we can see one has $z_2^0(\lambda = \lambda_0) = \nu_0$.

Let us end by noting that solutions to the eigenvalue problem are given by the solutions of (16) and (17) or by the solutions of the dispersion relation (31). Thus

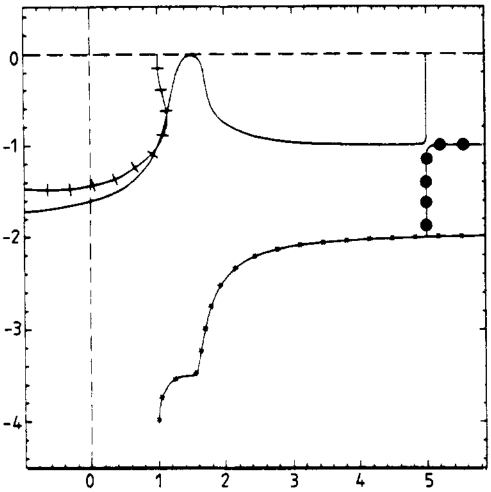

Figure 1. Trajectories of $\{z_i^0(\lambda);\ i=1,\ldots,4\}$ with $\omega_1^0=1$, $\omega_2^0=5$, $\alpha=4$ and $\beta=2$. –+–+– denotes z_1^0, —— denotes z_2^0, –×–×– denotes z_3^0 and –●–●– denotes z_4^0.

there is a connection between these equations. Indeed, using (5), we obtain ($k = 1,\ldots,N$)

$$\sigma_{kN}(\nu)=\sum_{j=1}^{N-1}(\hat{\sigma}_{kj}(\nu)+w_j(\nu)\sigma_{kj}(\nu)) \tag{36}$$

and $\Delta(\nu)$ can be written in the form (change column N by column $N-\sum_{j=1}^{N-1} w_j(\nu)$ column j)

$$\Delta(v)=\det\begin{bmatrix} \eta_{11}(\nu) & \cdots & \eta_{1,N-1}(\nu) & -(\omega_1^0-\nu)\omega_1(\nu)-\lambda^2\sum_{j=1}^{N-1}\hat{\sigma}_{1,j}(\nu) \\ \vdots & & \vdots & \vdots \\ \eta_{N-1,1}(\nu) & \cdots & \eta_{N-1,N-1}(\nu) & -(\omega_{N-1}^0-\nu)\omega_{N-1}(\nu)-\lambda^2\sum_{j=1}^{N-1}\hat{\sigma}_{N-1,j}(\nu) \\ \eta_{N,1}(\nu) & \cdots & \eta_{N,N-1} & \omega_N^0-\nu-\lambda^2\sum_{j=1}^{N}\hat{\sigma}_{N,j}(\nu) \end{bmatrix} \tag{37}$$

This relation shows that any solutions of (16) and (17) are also solutions of (31) but the converse is not true. Equations (16) and (17) thus give only some solutions to the eigenvalue problem.

I would like to thank A Grecos for suggesting this problem.

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